TYPE - IV :→ Eequation of the form

$$z = px + qy + f(p,q)$$

i,e clairaut's form.

or, Clairault's form (Alxis claude) (1713 - 1765).

changing p, q into a, b, we get the complete solution as

$$z = ax + by + f(a, b)$$

(Q) Find the compete integral of $z = px + qy + p^2 + q^2$

Sol^n:→ This is the form of $z = px + qy + f(p,q)$

∴ compete integral is

$$z = ax + by + a^2 + b^2$$

(Q) Find the singular integral of

$$z = px + qy + \log pq$$

Sol^n:→ It is of the form of $z = px + qy + f(p,q)$

∴ compete integral is

$$z = ax + by + \log(ab) \quad \text{— (1)}$$

S.I =

Differenceating (1) Partially w.r.to a & b

$$0 = x + \frac{1}{a}$$

$$0 = y + \frac{1}{b}$$

$$\therefore a = -\frac{1}{x} \quad \& \quad b = -\frac{1}{y}$$

Putting these values in (1)

$$Z = -1 - 1 + \log \frac{1}{xy}$$

$$\therefore Z = -2 - \log (x \cdot y)$$

(Q) $Z = px + qy - 2\sqrt{p \cdot q}$

Solⁿ:→ It is of the form of $Z = px + qy + f(p, q)$

$\therefore$ compete integral is $Z = ax + by - 2\sqrt{ab}$ — (1)

Singular integral:→ Differenceating (1) Partially w.r.to a & b.

$$0 = x - \frac{2}{2\sqrt{ab}} \cdot b$$

i,e $x = \sqrt{\frac{b}{a}}$ — (2)

& $0 = y - \frac{2}{2\sqrt{ab}} \cdot a$

i,e $y = \sqrt{\frac{a}{b}}$ — (3)

eliminating a, b the singular solⁿ is.

From (2) × (3)

$$x \cdot y = 1$$

(Q) $Z = px + qy + pq$

Solⁿ:→ It is of the form

$$Z = px + qy + f(p, q)$$

$\therefore$ compete integral is.

$$Z = ax + by + ab \quad - (1)$$

S.I :- Differenceating (1) Partially w.r.to a & b.

$$0 = x + b \qquad \therefore b = -x$$

$$0 = y + a \qquad a = -y$$

Substituing these value of a & b in (1)

$$Z = -xy - \cancel{xy} + \cancel{xy}$$

$$\therefore Z = -xy$$

(8) $(p-q)(z - px - qy) = 1$

Sol^n :- $z - px - qy = \frac{1}{p-q}$

$$\therefore z = px + qy + \frac{1}{p-q}$$

It is of the form of $z = px + qy + f(p,q)$

$\therefore$ complete integral is

$$Z = ax + by + \frac{1}{a-b} \quad - (1)$$

singular integral :-

(8) Solve $z = px + qy + c\sqrt{1+p^2+q^2}$

Sol^n:→ It is of the form of
$z = px + qy + f(p, q)$

∴ The compete integral is

$$z = ax + by + c\sqrt{1+a^2+b^2} \quad \text{— (1)}$$

Singular integral:→
Differenceating (1) w.r.to a & b we get

$$0 = x + \frac{ac}{\sqrt{1+a^2+b^2}} \quad \text{— (2)}$$

$$\& \quad 0 = y + \frac{bc}{\sqrt{1+a^2+b^2}} \quad \text{— (3)}$$

$$x^2 + y^2 = \frac{(a^2+b^2)c^2}{1+a^2+b^2}$$

∴ ~~1+ a²+b²~~ =

$$c^2 - x^2 - y^2 = c^2 - \frac{(a^2+b^2)c^2}{1+a^2+b^2}$$

$$= \frac{c^2 + (a^2+b^2)\cancel{x}c^2 - c^2(a^2+b^2)}{1+a^2+b^2}$$

$$\therefore c^2 - x^2 - y^2 = \frac{c^2}{1+a^2+b^2}$$

$$\therefore 1+a^2+b^2 = \frac{c^2}{c^2-x^2-y^2}$$

From ② & ③

$$a = \frac{x\sqrt{1+a^2+b^2}}{c} = \frac{-x}{\sqrt{c^2-x^2-y^2}}$$

$$b = \frac{y\sqrt{1+a^2+b^2}}{c} = \frac{-y}{\sqrt{c^2-x^2-y^2}}$$

Putting the value of a & b in ①

$$z = \frac{-x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$= \frac{c^2-x^2-y^2}{\sqrt{c^2-x^2-y^2}} = \sqrt{c^2-x^2-y^2}$$

or, $x^2+y^2+z^2 = c^2$